

Fredholm Approximations in the Scalar Bethe-Salpeter Equation

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The Fredholm approximation is discussed in the framework of the scalar Bethe-Salpeter equation. The trace of the angular momentum decomposed kernel is expressed in terms of Feynman parameter integrals which shows the relation to the vertex function. A new derivation for this representation is given which is far more direct than the previous one. Using this representation, several general features of the eigenvalues are discussed. For special cases, the trace is computed explicitly, and the numerical values are compared with the exact ones, obtained by variational methods.

1. Introduction

The Bethe-Salpeter (BS) equation can be solved analytically only for the Wick-Cutkosky model¹. In all cases of physical importance, one has to use numerical methods. Though the eigenvalues can be obtained with an arbitrary precision by variational methods², it is useful for some purposes to have approximate analytical solutions even if they are not so accurate as the numerical ones. The simplest of these solutions is the Fredholm (or trace) approximation. If the Fredholm determinants are expanded in powers of the coupling constant, the approximation consists in taking only the first order terms.

Since the kernel of the BS-equation is not square integrable due to the poles in the propagators, it is doubtful whether the Fredholm series will converge at all. For the simple ladder approximation, the kernel is of the Hilbert-Schmidt type after Wick rotation which is justified in the bound state region. So it might be not completely hopeless to give a proof for the unrotated equation as well.

There have been several attempts in the literature to prove the convergence. The proof given in³ is insufficient since it uses an unjustified simultaneous Wick rotation in multiple integrals. For the BS-equation for the on-shell T -matrix, the convergence has been established in⁴. Since the bound states show up as poles in the on-shell T -matrix, this proof is sufficient if one considers only the eigenvalues and not the eigenfunctions. This we will do in the present paper.

Numerical values for the bound state energies have been calculated by the Fredholm approxima-

tion in⁵ without decomposing into partial waves. In this case the trace of the kernel is logarithmically divergent, so one has to use the second iterated kernel. This difficulty is removed by the angular momentum decomposition. Since the solutions obtained in⁵ are not classified according to their angular momentum, it is impossible to compare them with the numerical solutions of² and⁶.

For the partial-wave BS-equation, a representation of the trace in terms of Feynman parametric integrals has been given in⁷. This representation shows the connection with the triangle graph of a φ^3 -theory. A much simpler derivation of this will be given in Section 3. Numerical values have not been calculated so far. This is done in Section 5. We find that the accuracy of the solutions depends heavily on the magnitude of the exchanged mass. In Sect. 4 we prove some general results on the behaviour of the eigenvalues calculated from the Fredholm approximation which are believed to be true also in the exact solutions, but have not been proven so far.

In recent years the Fredholm approximation has become quite fashionable in multiperipheral models. In this model one derives an integral equation for the absorptive part of scattering amplitude. From this one can do some sort of partial wave decomposition and arrives at an equation which is similar to the BS-equation in the ladder approximation. For a review see for example⁸. The Fredholm, or trace approximation as it is called there, has been considered by several authors in this context, see⁹ and earlier references cited there. So our investigations might be of some importance also in this context.

2. Partial Wave BS-Equation

In order to keep the paper self-contained and to fix the notation, we begin with a short derivation of

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the partial-wave BS-equation. We consider the BS-equation in the ladder approximation for two scalar particles having masses m_1 and m_2 which interact by the exchange of a third scalar particle of mass μ (all masses are to be understood with a small imaginary part) :

$$[m_1^2 - (p+k)^2][m_2^2 - (p-k)^2]\psi(p) = \frac{\lambda}{i\pi^2} \int d^4q \frac{\psi(q)}{\mu^2 - (p-q)^2} \quad (2.1)$$

where p is the relative momentum, $2k$ the total momentum and $\psi(p)$ the wave function. (2.1) is an eigenvalue problem for the coupling constant λ with the bound state mass $s = 4k^2$ acting as a parameter. We will always use the rest system $k = (k_0, 0, 0, 0)$.

Since (2.1) is invariant under three-dimensional rotations, its solutions can be classified with respect to their angular momentum l . Putting $p_1 = |\mathbf{p}|$ we make the ansatz

$$\psi(p) = \frac{1}{p_1} \psi_l(p_0, p_1) Y_{lm}(\vartheta, \varphi). \quad (2.2)$$

For the decomposition of (2.1) we use the addition theorem for the Legendre function

$$P_l(\cos \omega) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\vartheta, \varphi) Y_{lm}^*(\vartheta', \varphi') \quad (2.3)$$

with

$$\cos \omega = \cos \vartheta \cos \vartheta' + \sin \vartheta \sin \vartheta' \cos(\varphi - \varphi') \quad (2.4)$$

and the formula

$$(\beta - \cos \omega)^{-1} = \sum_{l=0}^{\infty} (2l+1) Q_l(\beta) P_l(\cos \omega). \quad (2.5)$$

The $Q_l(\beta)$ are the Legendre functions of the second kind, defined by:

$$Q_l(\beta) = \frac{1}{2} \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{\beta - \zeta}. \quad (2.6)$$

Defining

$$\beta = \frac{\mu^2 + p_1^2 + q_1^2 - (p_0 - q_0)^2}{2p_1 q_1} \quad (2.7)$$

the interaction kernel in (2.1) can be decomposed using (2.3), (2.5) :

$$\frac{1}{\mu^2 - (p-q)^2} = \frac{2\pi}{p_1 q_1} \sum_{l=0}^{\infty} \sum_{m=-l}^l Q_l(\beta) Y_{lm}(\vartheta, \varphi) Y_{lm}^*(\vartheta', \varphi') \quad (2.8)$$

where (ϑ, φ) , (ϑ', φ') are the polar angles of \mathbf{p} and \mathbf{q} , respectively. Now it is easy to derive the partial wave equation for the functions $\psi_l(p_0, p_1)$:

$$[m_1^2 - (p+k)^2][m_2^2 - (p-k)^2]\psi_l(p_0, p_1) = \frac{2\lambda}{\pi i} \int_0^{\infty} dq_1 \int_{-\infty}^{\infty} dq_0 Q_l(\beta) \psi_l(q_0, q_1). \quad (2.9)$$

Note that the inverse propagators on the left-hand side do not depend on the polar angles in the rest frame.

We are going to solve (2.9) in first Fredholm approximation. That means we will calculate the coupling constant $\lambda = \lambda_F$ from

$$1 = \lambda_F \sigma_l \quad (2.10)$$

where σ_l is the trace of the kernel :

$$\sigma_l = \frac{2}{\pi i} \int_0^{\infty} dp_1 \int_{-\infty}^{\infty} dp_0 \frac{Q_l(1 + \mu^2/2p_1^2)}{[m_1^2 + p_1^2 - (p_0 + k_0)^2][m_2^2 + p_1^2 - (p_0 - k_0)^2]}. \quad (2.11)$$

3. Relation to the Triangle Graph

In this section we will express the trace (2.10) by a Feynman parametric integral which will show the correspondance to the triangle graph. The final result (3.17) has already been given in⁷ but our derivation is much more direct.

We start by replacing the Q_l -function by a Legendre polynomial by using (2.6) :

$$\sigma_l = \frac{2}{\pi i} \int_{-1}^1 d\zeta P_l(\zeta) \int_0^{\infty} dp_1 \int_{-\infty}^{\infty} dp_0 \frac{p_1^2}{\mu^2 + 2(1-\zeta)p_1^2} \frac{1}{m_1^2 + p_1^2 - (p_0 + k_0)^2} \frac{1}{m_2^2 + p_1^2 - (p_0 - k_0)^2}. \quad (3.1)$$

Next we combine the three denominators by introducing Feynman parameter x_1 , x_2 , and x_3 . With the abbreviations

$$\alpha = m_1^2 x_1 + m_2^2 x_2 + \mu^2 x_3 - k_0^2 (x_1 + x_2), \quad \beta = x_1 + x_2 + 2(1 - \zeta) x_3, \quad \gamma = 2k_0(x_2 - x_1), \quad \delta = x_1 + x_2 \quad (3.2)$$

we get the following expression:

$$\sigma_l = \frac{4}{\pi i} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1 - x_1 - x_2 - x_3) \int_{-1}^1 d\zeta P_l(\zeta) \int_0^\infty dp_1 \int_{-\infty}^\infty dp_0 p_1^2 \{ \alpha + \beta p_1^2 + \gamma p_0 - \delta p_0^2 \}^{-3}. \quad (3.3)$$

Now we will calculate the integral with respect to p_0 and p_1 . This can be done by rather common methods. Defining

$$I = \frac{1}{i} \int_0^\infty dp_1 \int_{-\infty}^\infty dp_0 p_1^2 \{ \alpha + \beta p_1^2 + \gamma p_0 - \delta p_0^2 \}^{-3} \quad (3.4)$$

we first translate p_0 to $p_0 - \gamma/2\delta$ and define $\alpha' = \alpha + \gamma^2/4\delta$. Then we rotate the integration path of p_0 counterclockwise to the imaginary axis. This is possible because of the imaginary parts in the masses m_1 and m_2 . So we have:

$$I = \int_0^\infty dp_1 \int_{-\infty}^\infty dp_0 p_1^2 \{ \alpha' + \beta p_1^2 + \delta p_0^2 \}^{-3}. \quad (3.5)$$

By the symmetry of the integrand, the p_1 integration can be extended to the whole axis. Next we perform a scale transformation so that the denominator will only depend on $p_0^2 + p_1^2$. Then we introduce spherical coordinates in the p_0, p_1 -plane by $p_0 = p \cos \vartheta$, $p_1 = p \sin \vartheta$. So we have finally:

$$I = \frac{1}{2\sqrt{\beta^3\delta}} \int_0^\infty dp \int_0^{2\pi} d\vartheta \frac{p^3 \sin^2 \vartheta}{(\alpha' + p^2)^3}. \quad (3.6)$$

The integration is now trivial and yields:

$$I = \frac{\pi}{8\alpha'\sqrt{\beta^3\delta}}. \quad (3.7)$$

If we substitute this into (3.3) and use the abbreviations (3.2) we get:

$$\sigma_l = \frac{1}{2} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1 - x_1 - x_2 - x_3) \sqrt{x_1 + x_2} \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{(x_1 + x_2 + 2(1 - \zeta)x_3)^{3/2}} \cdot \{ m_1^2 x_1 + m_2^2 x_2 + \mu^2 x_3 - m_1^2 x_1 x_3 - m_2^2 x_2 x_3 - \mu^2 x_3^2 - s x_1 x_2 \}^{-1}. \quad (3.8)$$

Before we do the ζ -integration we perform a transformation on the Feynman parameters by:

$$x_1 = x_1' \frac{1 - x_3'}{1 - x_3' + x_3'^2}, \quad x_2 = x_2' \frac{1 - x_3'}{1 - x_3' + x_3'^2}, \quad x_3 = \frac{x_3'}{1 - x_3' + x_3'^2}. \quad (3.9)$$

Dropping the primes on the new variables, the trace now reads

$$\sigma_l = \frac{1}{2} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1 - x_1 - x_2 - x_3) \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{(1 - 2x_3\zeta + x_3^2)^{3/2}} \cdot (1 - x_3^2) \{ m_1^2 x_1 + m_2^2 x_2 + \mu^2 x_3 - m_1^2 x_1 x_3 - m_2^2 x_2 x_3 - s x_1 x_2 \}^{-1}. \quad (3.10)$$

The purpose of the transformation of the Feynman parameters is now seen from the ζ -integrand: The denominator is now the cube of the generating function for the Legendre polynomials. So one may hope to do the integral in a simple manner. This is really true as can be seen in the following way: We define:

$$I_l(x) = \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{(1 - 2x\zeta + x^2)^{3/2}}. \quad (3.11)$$

From the recursion relation for the Legendre polynomials

$$(l+1)P_{l+1}(\zeta) + lP_{l-1}(\zeta) = (2l+1)\zeta P_l(\zeta) \quad (3.12)$$

we have the relation:

$$(l+1)I_{l+1}(x) + lI_{l-1}(x) = (2l+1) \left[x \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{(1-2x\zeta+x^2)^{3/2}} + \frac{d}{dx} \int_{-1}^1 d\zeta \frac{P_l(\zeta)}{(1-2x\zeta+x^2)^{1/2}} \right]. \quad (3.13)$$

The second integral on the right-hand side can now be done by using the fact that $(1-2x\zeta+x^2)^{-1/2}$ is the generating function for the Legendre polynomials, and the orthogonality of the $P_l(\zeta)$, yielding:

$$\int_{-1}^1 d\zeta P_l(\zeta)/(1-2x\zeta+x^2)^{1/2} = 2x^l/(2l+1). \quad (3.14)$$

So we have from (3.13) the recursion relation:

$$(l+1)I_{l+1}(x) + lI_{l-1}(x) = (2l+1)xI_l(x) + 2lx^{l-1}. \quad (3.15)$$

$I_0(x)$ and $I_1(x)$ can be computed by substituting the explicit formulas for $P_0(\zeta)$ and $P_1(\zeta)$ into (3.11). Using them as the starting points for the recursion, we get from (3.15) the general result:

$$I_l(x) = 2x^l/(1-x^2). \quad (3.16)$$

Substituting this into (3.10) we have the desired Feynman parametric integral representation of the trace:

$$\sigma_l = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1-x_1-x_2-x_3) x_3^l \cdot \{m_1^2 x_1 + m_2^2 x_2 + \mu^2 x_3 - m_1^2 x_1 x_3 - m_2^2 x_2 x_3 - s x_1 x_2\}^{-1}. \quad (3.17)$$

A similar integral representation is obtained for the triangle graph. For completeness, we will derive it following¹⁰. If the external momenta are p_1 , p_2 , and p_3 and the masses of the internal lines are m_1^2 , m_2^2 , and m_3^2 , the Feynman integral for the triangle graph is:

$$I = \int d^4q \{ [q^2 - m_1^2] [(q+p_3)^2 - m_2^2] [(q-p_2)^2 - m_3^2] \}^{-1}. \quad (3.18)$$

Introducing the parameters x_1 , x_2 , and x_3 we can do the q -integration by standard methods and get:

$$I = -i\pi^2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1-x_1-x_2-x_3) \cdot \{x_2 x_3 p_1^2 + x_1 x_3 p_2^2 + x_1 x_2 p_3^2 - x_1 m_1^2 - x_2 m_2^2 - x_3 m_3^2\}^{-1}. \quad (3.19)$$

If we now restrict p_1 and p_2 to their mass-shell values $p_1^2 = m_2^2$, $p_2^2 = m_1^2$ we obtain by defining $s = p_3^2$, $\mu^2 = m_3^2$ the same expression as in (3.17) apart from the factor x_3^l . Hence for $l=0$ the trace of the BS-kernel is given by the amplitude of the triangle graph with two external lines on-shell.

4. General Properties of the Eigenvalues

From the integral representation (3.17) we can deduce several general properties of the eigenvalues, obtained by the Fredholm approximation. For brevity we denote the denominator in (3.17) by D . So we have

$$\sigma_l = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(1-x_1-x_2-x_3) x_3^l D^{-1}. \quad (4.1)$$

If we restrict us to positive values of the angular momentum, the singularities are determined by D . For the denominator we have the estimate:

$$D \geq x_1 x_2 [(m_1 + m_2)^2 - s] + x_3 \mu^2 \quad (4.2)$$

and in the bound state region $s < (m_1 + m_2)^2$ we have therefore $D \geq 0$. In the following we will always consider this energy region. If we have $\mu^2 > 0$ we have furthermore $D > 0$. In the Wick-Cutkosky model with $\mu^2 = 0$ the denominator has a zero at $x_1 = x_2 = 0$, and the integral is logarithmically divergent. So the coupling constant is $\lambda_F = 0$ which is a quite bad approximation for the exact values¹.

From $D > 0$ we have $\sigma_l > 0$, and from (2.10) also $\lambda_F > 0$. For the equal mass case $m_1^2 = m_2^2$, after Wick rotation the BS-kernel is selfadjoint and positive

definite¹¹. So by well-known theorems we have a real and positive spectrum and at least one eigenvalue. In the Fredholm approximation, these features hold also in the unequal mass case.

Next we investigate the relation between s and λ_F . If s is increased, D will decrease but remain positive for s below the elastic threshold. So σ_l will increase, and λ_F decrease. So we have

$$d\lambda_F/ds < 0 \quad (4.3)$$

(4.3) had to be expected because of physical reasons: For a stronger interaction the binding energy will become larger and hence the mass of the bound state will go down. For the exact eigenvalues, (4.3) has been proven only in special cases¹².

From the Fredholm approximation we get one eigenvalue λ_F . Because of (4.3) there corresponds to it exactly one bound state mass s . This is only true for positive values of the angular momentum, even below the elastic threshold. For $l < 0$ we have a singularity at $x_3 = 0$ in (4.1). So we cannot continue analytically to the left half of the complex angular momentum plane. For a detailed discussion of this point see⁹.

For positive values of l , we can get from (4.1) the slope of Regge trajectories. The explicit formula

turns out to be rather useless since the integrals are too involved. But we can easily prove that the trajectories rise in the Chew-Frautschi plot: Since $x_3 \leq 1$, we have $x_3^{l+1} D^{-1} \leq x_3^l D^{-1}$ and therefore $\sigma_{l+1} < \sigma_l$. So we get $\lambda_{F,l+1} > \lambda_{F,l}$ from (2.10). If we require the same coupling constant for all angular momenta, we get from (4.3) $s_{l+1} > s_l$. So we have:

$$dl/ds > 0. \quad (4.4)$$

If we restrict ourselves to the equal mass case, the BS-kernel can be transformed into a positive definite kernel of the Hilbert-Schmidt type. Therefore we have at least one eigenvalue λ_1 which we take to be the smallest one in the case of several eigenvalues. We can now apply Mercer's theorem¹³:

$$\sigma = \sum_i \lambda_i^{-1} \quad (4.5)$$

where the sum runs over all eigenvalues. Isolating λ_1 from the sum, we have from (2.10) and (4.5):

$$\lambda_F^{-1} = \sigma = \lambda_1^{-1} + \sum_{i \neq 1} \lambda_i^{-1}. \quad (4.6)$$

Since $\lambda_i > 0$ for positive definite kernels we get the estimate $\lambda_F \leq \lambda_1$. By the Fredholm approximation we get therefore a lower bound for the exact eigenvalues. Numerically this turns out to be true also in the unequal mass case.

5. Numerical Results

We now turn to the evaluation of the parameter integrals. For $s < (m_1 + m_2)^2$ we have $D > 0$, and we can drop the imaginary parts of the masses because there are no poles. In the scattering region we have to be more careful. With the parameter transformation

$$x_2 = x - y, \quad x_3 = 1 - x \quad (5.1)$$

two of the integrations can be carried out rather elementary for arbitrary values of l , yielding:

$$\sigma_l = \int_0^1 dx (1-x)^l \{ x^2 [s - (m_1 + m_2)^2] [s - (m_1 - m_2)^2] - 4 s \mu^2 (1-x) \}^{-1/2} \cdot \ln \frac{2 \mu^2 (1-x) + x^2 (m_1^2 + m_2^2 - s) - x \{ x^2 [s - (m_1 + m_2)^2] [s - (m_1 - m_2)^2] - 4 s \mu^2 (1-x) \}^{1/2}}{2 \mu^2 (1-x) + x^2 (m_1^2 + m_2^2 - s) + x \{ x^2 [s - (m_1 + m_2)^2] [s - (m_1 - m_2)^2] - 4 s \mu^2 (1-x) \}^{1/2}}. \quad (5.2)$$

As can be seen, the last integration is very complicated. For the case $l=0$, i. e. the triangle graph, it was carried out explicitly in¹⁴. It involves the sum of sixteen dilogarithms (for the dilogarithm see for example¹⁵). Since the dilogarithm $\mathcal{L}_2(x)$ is essentially a generalized hypergeometric function:

$$\mathcal{L}_2(x) = x {}_3F_2(1, 1, 1; 2, 2; x) \quad (5.3)$$

we expect for other values of l the dilogarithms to be replaced by $x {}_3F_2(1, l+1, l+1; l+2, l+2; x)$ as in the simpler case considered below. We hope to prove this in the future.

Here we confine us to some special cases where the integration is much simpler. The simplest one is the equal mass case with vanishing bound state mass: $m_1 = m_2 = 1$, $s = 0$. Then we have:

$$\sigma_l = \int_0^1 dx (1-x)^l \int_0^x dy \{ (1-x)\mu^2 + x^2 \}^{-1}. \quad (5.4)$$

If we decompose the denominator into partial fractions we can use the integral

$$\int_0^1 dx x^l / (x - x_1) = -\frac{1}{l+1} \frac{1}{x_1} {}_2F_1(1, l+1; l+2; x_1^{-1}) \quad (5.5)$$

which is found in¹⁶. For integer values of l , the hypergeometric function ${}_2F_1$ can be expressed in terms of elementary functions:

$${}_2F_1(1, l+1; l+2; x) = -(l+1)x^{-l-1} \left\{ \ln(1-x) + \sum_{n=1}^l n^{-1} x^n \right\}. \quad (5.6)$$

With $x_{1,2} = -\frac{1}{2}(\mu^2 - 2) \pm \frac{1}{2}\mu\sqrt{\mu^2 - 4}$ we have finally for the trace:

$$\sigma_l = \frac{1}{\mu\sqrt{\mu^2 - 4}} \left\{ x_1^l (1-x_1) \ln \frac{x_1-1}{x_1} - x_2^l (1-x_2) \ln \frac{x_2-1}{x_2} + (1-x_1) \sum_{n=1}^l \frac{x_1^{l-n}}{n} - (1-x_2) \sum_{n=1}^l \frac{x_2^{l-n}}{n} \right\}. \quad (5.7)$$

For $l=0$ this simplifies considerably:

$$\sigma_0 = \frac{\mu}{\sqrt{4-\mu^2}} \arccos \frac{\mu}{2} - \ln \mu. \quad (5.8)$$

For $\mu > 2$ we have to replace $\arccos \mu/2$ by $\text{Arcosh } \mu/2$. (5.8) has been obtained already in¹⁷.

From (5.8) we can calculate the coupling constants λ_F for different values of the exchanged mass μ . The results are summarized in the following table. The exact values λ_1 are taken from⁶.

μ	λ_F	λ_1
0.25	0.638	2.214
0.5	0.968	2.566
0.7	1.235	2.888
1.0	1.656	3.418
1.25	2.024	3.896
2.5	4.18	6.68

Table 1. Approximate (λ_F) and exact (λ_1) eigenvalues for different values of the exchanged mass μ .

From the table we can see several facts: First we see that indeed $\lambda_F < \lambda_1$ as was shown in section 4. Secondly λ_F is a better approximation for higher values of the exchange mass μ . This can be under-

stood in the following way: In coordinate space in the nonrelativistic limit, the exchange of a mass μ scalar particle is equivalent to the Yukawa potential $V(r) \sim e^{-\mu r}/r$. For an increasing mass μ which is the range parameter in $V(r)$, the potential becomes more and more similar to a δ -function. So we can write $\lim_{\mu \rightarrow \infty} e^{-\mu r}/r \sim \delta(r)$. But for the contact potential $V(r) \sim \delta(r)$, the Fredholm approximation is the exact solution. So the behaviour of the eigenvalues is explained.

For $l=1$ we have only calculated the trace for $\mu=1$. We find $\sigma_1 = -1 + 2\pi/3\sqrt{3}$ yielding the eigenvalue $\lambda_F = 4.78$ whereas the exact value is $\lambda_1 = 16.4$, again taken from⁶. The approximation is much worse than for $l=0$. This is understood from the fact that for $l \neq 0$ we have the centrifugal barrier as an additional long-range repulsive potential.

The variation of λ_F with the bound state mass s can be calculated in a relatively simple manner if one of the external masses is zero. Putting $m_2=0$, $l=0$, two integrations in (3.17) are trivial, and the third one can be done by using the integrals listed in¹⁵. The result is:

$$\sigma_0 = \frac{1}{s-m_1^2} \left\{ 2 \mathcal{L}_2 \left(\frac{\mu m_1}{\sqrt{(s-m_1^2)^2 + s\mu^2}}, \Theta \right) - 2 \mathcal{L}_2 \left(\frac{m_1(\mu^2 + s - m_1^2)}{\mu \sqrt{(s-m_1^2)^2 + s\mu^2}}, \Theta \right) + \mathcal{L}_2 \left(\frac{\mu^2 + s - m_1^2}{\mu^2} \right) - \mathcal{L}_2 \left(\frac{s(\mu^2 + s - m_1^2)}{(s-m_1^2)^2 + s\mu^2} \right) - \mathcal{L}_2 \left(\frac{s\mu^2}{(s-m_1^2)^2 + s\mu^2} \right) - \frac{\pi^2}{6} \right\} \quad (5.9)$$

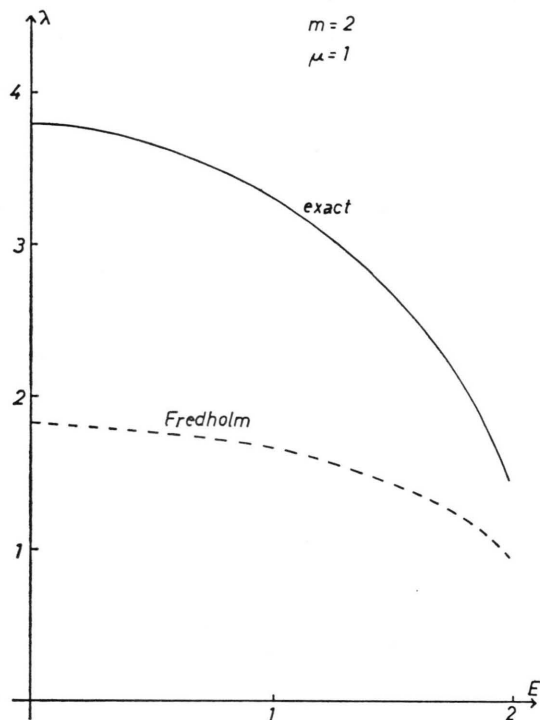


Fig. 1. Eigenvalues λ plotted against $E = \sqrt{s}$ for $m_1=2$, $m_2=0$, $\mu=1$. The dashed line gives the eigenvalues obtained from Fredholm approximation whereas the solid one gives the exact eigenvalues.

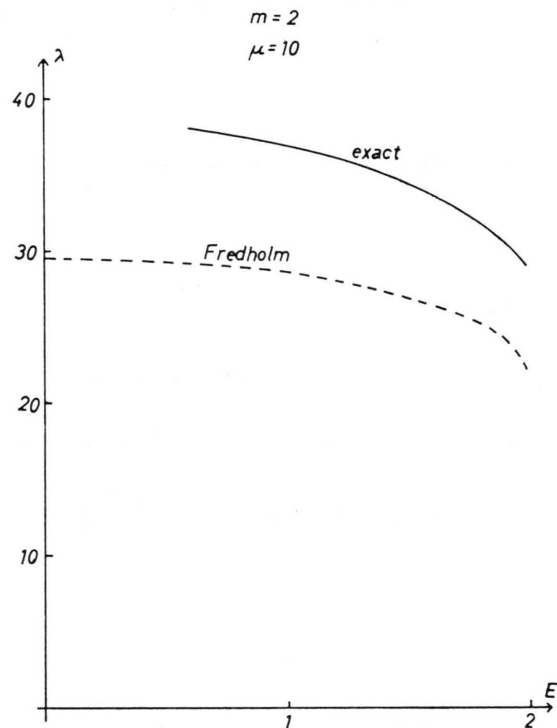


Fig. 2. Same as Fig. 1, but with $\mu=10$. Notice the different scale of the λ -axis.

with the angle Θ defined by

$$\cos \Theta = \frac{\mu(s + m_1^2)}{2 m_1 \sqrt{(s - m_1^2)^2 + s \mu^2}}. \quad (5.10)$$

For $\cos \Theta > 1$, one has to modify (5.9) a little bit. $\mathcal{L}_2(x)$ denotes the dilogarithm:

$$\mathcal{L}_2(x) = -\int_0^x dy \ln(1-y)/y \quad (5.11)$$

and $\mathcal{L}_2(x, \Theta)$ the real part of the dilogarithm of the complex argument $z = x e^{i\Theta}$

$$\mathcal{L}_2(x, \Theta) = -\frac{1}{2} \int_0^x dy \frac{\ln(1 - 2y \cos \Theta + y^2)}{y}. \quad (5.12)$$

For $m_2=2$, $\mu=1$ and $\mu=10$ the eigenvalues are plotted in Figure 1 and 2, respectively, and compared with the exact values from⁶. We again find the better agreement for the higher exchanged mass, and the approximate eigenvalues lie below the exact

ones. Furthermore the approximation is better for weakly bound states, that means for smaller values of the coupling constant. This is understandable since we have taken into account only the lowest order term in the expansion of the Fredholm determinant, and all higher order terms are suppressed for small coupling constants.

Summarizing the results, we see that the Fredholm approximation reproduces some general features of the exact solutions, but the numerical accuracy of the eigenvalues is much lower than of those obtained by variational methods². The computation of higher order terms in the Fredholm series cannot be carried out because of the complexity of the integrals involved.

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